INVESTIGATING THE EFFECT OF INHOMOGENEOUS RESISTIVITY ON BULK RRR AND HEAT CONDUCTIVITY USING A LATTICE GREEN'S FUNCTIONS METHOD

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Abstract

A method was developed to calculate the bulk RRR (residual resistivity ratio) which would be measured on a superconducting cavity or sample with an inhomogeneous resistivity and arbitrary geometry. The method involves modelling the object as a network of resistors and employs lattice Green's functions. A further adaptation of the method which allows the heat transport characteristics of such an object to be predicted is also described.

INTRODUCTION

The RRR of a metal is defined as the ratio of the resistivity at 300 K to the resistivity at a low temperature just above the superconducting transition (in this paper 'low temperature' is defined as 10 K) as shown in Eq. (1) where $\rho(T)$ is the resistivity at temperature T [1].

$$\mathrm{RRR} = \frac{\rho(300\,\mathrm{K})}{\rho(10\,\mathrm{K})} \tag{1}$$

As explained below, the RRR is related to the thermal conductivity via the Weidemann-Franz law [2] [3] as shown in Eq. (2), where κ is the thermal conductivity, σ is the electrical conductivity, T is the temperature and $\mathcal{L}$ is a constant of proportionality known as the Lorentz number.

$$\frac{\kappa}{\sigma} = \mathcal{L}T \tag{2}$$

As a metal with a higher RRR has a higher electrical conductivity at low temperatures, so too will it have a higher thermal conductivity. Therefore manufacturing bulk Niobium cavities from material with a high RRR is desirable as the increased thermal conductivity will: lead to a cooler inside surface, increase the Q_0 (intrinsic quality factor) for a given field level and suppress thermal breakdown allowing higher field levels to be reached [4].

The widespread use of Eq. (1), as measured on a sample, as a predictor for the heat carrying capacity of a cavity is based on the implicit assumption that the resistivity within the material comprising the cavity is, to a good approximation, homogeneous and identical to that of the sample. However, there are many processes during the fabrication and preparation of a cavity which are able to locally affect the resistivity including: electron beam welding, mechanical working of the material and diffusion of impurities during heat treatment [5].

Such processes have the potential to cause the RRR measured using a four-wire technique [6] to differ for objects of different geometries. Previously presented measurements detailed a case where a 650 °C heat treatment had a greater detrimental effect on the bulk RRR of Niobium samples than on a cavity even though both had been made from the same sheet of Niobium and undergone the heat treatment in the same oven at the same time [7].

The measured RRR and heat transport capability of a cavity may not be affected in the same way by an inhomogeneous resistivity. For example, consider a four-wire RRR measurement of an elliptical cavity with one voltage and current probe on either side of its equator. If it is the case that the weld causes an increase in low temperature resistivity in the vicinity of the weld [8] [9], than a weld along the equator of the cavity would likely cause a greater proportional reduction in the measured RRR, where all the current must flow across the weld, than on the average heat transport capability, where the path of heat flow from the inside to the outside surface largely avoids the weld.

In the method here described, the cavity walls or sample are modelled as a grid of nodes connected by a network of resistors as shown in Fig. 1. Following a similar method to that shown in [10] an expression is obtained for a resistance defined as:

$$R^{\mu\nu}_{\alpha\beta} = \frac{V_\alpha - V_\beta}{I} \tag{3}$$

where V_i is the voltage at any node i and I is the magnitude of current which flows into the network at node μ and out of the network at node ν. From this expression the RRR of a sample or cavity that would be measured with a four-wire technique can be predicted provided the resistivity distribution and temperature dependence can be modelled. Alternatively the validity of a proposed resistivity distribution can be tested by comparison of the measured and calculated RRR.

LATTICE GREEN FUNCTION METHOD FOR RRR

Kirchoff's Law states that:

$$\sum_{j=1}^{N}{}' c_{ij}\left(V_i - V_j\right) = I_i \tag{4}$$

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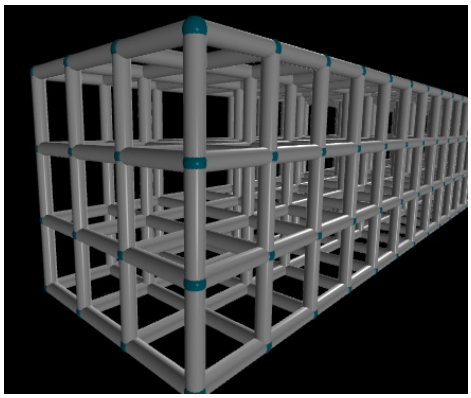

Figure 1: Modelling of a sample as a network of nodes connected by resistors.

where V_i and I_i are respectively, at node i: the voltage, and the current flowing into the network. The prime indicates the omission of the term $j = i$ in the summation. c_{ij} is defined as:

$$c_{ij} = \frac{1}{r_{ij}} = \frac{A_{ij}\sigma_{ij}}{l_{ij}} \tag{5}$$

where r_{ij}, A_{ij}, σ_{ij} and l_{ij} are respectively: the resistance, cross sectional area, conductivity and length, of the edge connecting nodes i and j.

Equation (4) can be written as:

$$\mathbf{L}\vec{V} = \vec{I} \tag{6}$$

where I_i is the current flowing into the network at node i, V_i is the voltage at node i, and:

$$\mathbf{L} = \begin{pmatrix} c_1 & -c_{12} & \cdots & -c_{1N} \\ -c_{21} & c_2 & \cdots & -c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{N1} & -c_{N2} & \cdots & c_N \end{pmatrix} \tag{7}$$

with:

$$c_i \equiv \sum_{j=1}^{N}{}' c_{ij} \tag{8}$$

$\mathbf{L}$ has eigenvectors $\vec{\Psi}_i$ and eigenvalues λ_i for $i = 1, 2, \ldots, N$. The components of $\vec{\Psi}_i$ are $\Psi_{i\alpha}$ for $\alpha = 1, 2, \ldots N$. It is easily verified that $\mathbf{L}$ has a normalised eigenvector $\vec{\Psi}_1$ where:

$$\Psi_{1\alpha} = \frac{1}{\sqrt{N}} \quad \forall\alpha \tag{9}$$

with corresponding eigenvalue $\lambda_1 = 0$. As:

$$|\mathbf{L}| = \prod_{i=1}^{N} \lambda_i \tag{10}$$

for any matrix, $|\mathbf{L}| = 0$, and so it is not possible to find $\mathbf{L}^{-1}$ in order to directly solve Eq. (6) for $\vec{V}$. Physically this is related to the fact that infinitely many solutions for $\vec{V}$ can be found by adding the same voltage to every node in the network. This is not a fundamental limitation as it is always the difference in voltage between two nodes which is of interest, or identically the voltage of a node once the voltage of any other node has been assigned. Mathematically the analysis is continued by considering the matrix $\mathbf{L}(\epsilon)$ where:

$$\mathbf{L}(\epsilon) = \mathbf{L} + \epsilon\mathbf{I} \tag{11}$$

$\mathbf{I}$ is the $N \times N$ identity matrix and ϵ is some number.

It is clear from Eq. (11) that $\mathbf{L}(\epsilon)$ has the same eigenvectors as $\mathbf{L}$ and that the corresponding eigenvalues $\lambda_i(\epsilon) = \lambda_i + \epsilon \quad \forall i$. As $\mathbf{L}$ is a symmetric matrix it can be diagonalised by an orthogonal matrix $\mathbf{O}$ whose columns are the normalised eigenvectors of $\mathbf{L}$:

$$\mathbf{O}^T\mathbf{L}\mathbf{O} = \mathbf{D} \tag{12}$$

where $\mathbf{D}$ is the diagonal matrix whose elements are:

$$D_{ij} = \delta_{ij}\lambda_i \tag{13}$$

with δ_{ij} the Dirac delta function. From Eq. (11) and Eq. (12) it is simple to show that:

$$\mathbf{O}^T\mathbf{L}(\epsilon)\mathbf{O} = \mathbf{D}(\epsilon) \tag{14}$$

where $\mathbf{O}$ is the same orthogonal matrix in Eq. (12) explicitly:

$$O_{ij} = \Psi_{ji} \tag{15}$$

and $\mathbf{D}(\epsilon)$ is the diagonal matrix with elements:

$$D(\epsilon)_{ij} = \delta_{ij}(\lambda_i + \epsilon) \tag{16}$$

From Eq. (14) the following expression for $\mathbf{L}^{-1}$ (the Green's function) can be found:

$$\mathbf{L}^{-1}(\epsilon) = \mathbf{O}\mathbf{D}^{-1}(\epsilon)\mathbf{O}^T \tag{17}$$

where the elements of $\mathbf{D}^{-1}(\epsilon)$ are given explicitly as:

$$D^{-1}(\epsilon)_{ij} = \delta_{ij}\left(\frac{1}{\lambda_i + \epsilon}\right) \tag{18}$$

Therefore from Eqs. (15)-(18) the following expression for the elements of $\mathbf{L}^{-1}(\epsilon)$ can be obtained:

$$L^{-1}(\epsilon)_{\alpha\beta} = \sum_{i=1}^{N} \frac{\Psi_{i\alpha}\Psi_{i\beta}}{\lambda_i + \epsilon} \tag{19}$$

By using Eq. (9), Eq. (19) can also be written as:

$$L^{-1}(\epsilon)_{\alpha\beta} = \frac{1}{N\epsilon} + g_{\alpha\beta}(\epsilon) \tag{20}$$

where:

$$g_{\alpha\beta}(\epsilon) = \sum_{i=2}^{N} \frac{\Psi_{i\alpha}\Psi_{i\beta}}{\lambda_i + \epsilon} \tag{21}$$

In close accordance with Eq. (6) it is now written that:

$$\vec{V}(\epsilon) = \mathbf{L}^{-1}(\epsilon)\vec{I}(\epsilon) \tag{22}$$

By substitution of Eq. (20) into Eq. (22) the following explicit equation for the components of $\vec{V}(\epsilon)$ is obtained:

$$\vec{V}(\epsilon)_i = \sum_{j=1}^{N} \frac{I_j}{N\epsilon} + \sum_{j=1}^{N} g_{ij}(\epsilon) I_j \tag{23}$$

for the case where all the impedances in the network are purely real (which the preceding mathematics has already assumed anyhow), we must have that:

$$\sum_{j=1}^{N} I_j = 0 \tag{24}$$

From Eq. (24), Eq. (23) becomes:

$$\vec{V}(\epsilon)_i = \sum_{j=1}^{N} g_{ij}(\epsilon) I_j \tag{25}$$

It is further assumed that all the current flows into the network at node μ and leaves the network at node ν such that:

$$I_i = I(\delta_{i\mu} - \delta_{i\nu}) \tag{26}$$

By taking the limit $\epsilon \rightarrow 0$, defining $g_{ij} = g_{ij}(\epsilon = 0)$, and substituting in Eq. (26), Eq. (25) becomes:

$$\vec{V}_i = I(g_{i\mu} - g_{i\nu}) \tag{27}$$

Finally, by substituting in Eq. (27) the final expression for R as defined in Eq. (3) becomes:

$$R^{\mu\nu}_{\alpha\beta} = g_{\alpha\mu} + g_{\beta\nu} - g_{\alpha\nu} - g_{\beta\mu} \tag{28}$$

LATTICE GREEN FUNCTION METHOD FOR HEAT CONDUCTIVITY

The method outlined in the previous section is readily adapted to calculate an absolute thermal resistance $Z^{\mu\nu}_{\alpha\beta}$ defined as:

$$Z^{\mu\nu}_{\alpha\beta} = \frac{T_\alpha - T_\beta}{P_{th}} \tag{29}$$

where T_i is the temperature at node i in the steady state and P_{th} is the heat entering the network at node μ and leaving at node ν per unit time in the steady state.

In effect instead of solving the equation:

$$J_I = -\sigma \nabla V \tag{30}$$

on a network (where J_I is the current density) one would now need to solve the mathematically identical equation:

$$J = -\kappa \nabla T \tag{31}$$

where J is the heat flux, κ is the thermal conductivity and T is the temperature. As such the elements c_{ij} which were defined in Eq. (5) must now be defined as follows:

$$c_{ij} = \frac{A_{ij}\kappa_{ij}}{l_{ij}} \tag{32}$$

where κ_{ij} is the thermal conductivity in the region surrounding an edge connecting two nodes (and zero if the nodes are not connected) A_{ij} is the cross sectional area associated with that edge and l_{ij} is the length of the edge.

The definitions of c_i, $\mathbf{L}$ and $g_{\alpha\beta}$ remain unchanged and the expression for $Z^{\mu\nu}_{\alpha\beta}$ is simply:

$$Z^{\mu\nu}_{\alpha\beta} = g_{\alpha\mu} + g_{\beta\nu} - g_{\alpha\nu} - g_{\beta\mu} \tag{33}$$

as before.

In order for $Z^{\mu\nu}_{\alpha\beta}$ to be of any practical use for predicting the thermal behaviour of an object, the network model used to calculate $R^{\mu\nu}_{\alpha\beta}$ needs to be adapted. This is because the assumption all the heat flows into the network at a single node and out of another is unlikely to be realistic. For illustrative purposes one possible adaptation for the case of a superconducting cavity operating in a liquid helium bath is briefly described.

First two artificial nodes are added to the network model, one node (θ) is connected to every 'real' node on the cavity's inner surface whilst the other (ϕ) is connected to every node on the cavities outer surface. By choosing $c_{\phi n}\, \forall\, n$, where n is any node on the cavity's outer surface, such that:

$$c_{\phi n} >> c_{ij} \quad \forall\, i, j \notin \{\theta, \phi\} \tag{34}$$

and by a judicious choice of $c_{\theta m}\, \forall\, m$, where m is any node on the cavity's outer surface, and by ensuring also that:

$$c_{\theta m} << c_{ij} \quad \forall\, i, j \notin \{\theta, \phi\} \tag{35}$$

the distribution of the power on the cavity surface can be made to match such a distribution obtained from any rf simulation software. Also, the temperature on the outer surface which is in contact with the helium bath will be uniform. Then $Z^{\theta\phi}_{nm}$ can be calculated from Eq. (33)and used to calculate the temperature at any point on the cavities inner surface for a given total power dissipation P_{th}.

SUMMARY AND FUTURE WORK

The methods described in this paper offer a new and powerful tool to predict the effect on cavity performance from any process during or after fabrication which causes local variations in the resistivity. However it is reliant on an accurate model of the resistivity distribution. As such work is under way to better understand the likely resistivity distribution. As part of this effort many new samples are being fabricated of different geometries, varying surface area to volume ratios, with different heat treatments and both with and without welds. The RRR for each of these will be measured with a four-wire technique to verify and improve the resistivity models which are being developed.

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